

# Is it possible to check urgently the 5-loop analytical results for the $e^+e^-$ -annihilation Adler function ?

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## ABSTRACT

Considering the results of recent distinguished analytical calculations of the 5-loop single-fermion loop corrections to the QED  $\beta$ -function we emphasize that to our point of view it is important to perform their independent cross-checks. We propose one of the ways of these cross-check. It is based on the application of the original Crewther relation. We derive the new analytical expressions for the  $C_F^4\alpha_s^4$ -contributions to the Bjorken polarized sum rule. If results of possible direct calculations will agree with the presented expression, then the appearance of  $\zeta_3$ -term in the 5-loop correction to the QED  $\beta$ -function and in the  $C_F^4\alpha_s^4$  contribution into the  $e^+e^-$  annihilation Adler function will get independent support and may be analysed within the framework of the recently introduced concept of “maximal transcendentality”.

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# 1 Introduction

Quite recently the complicated analytical expression for the non-singlet order  $\alpha_s^4$  contribution to the  $e^+e^-$  annihilation Adler function

$$D^{NS}(Q^2) = Q^2 \int_0^\infty \frac{R(s)}{(s+Q^2)^2} ds = 3 \sum_F Q_F^2 C_D^{NS}(a_s(Q^2)) = 3 \sum_F Q_F^2 \left[ 1 + \sum_{n=1}^{n=4} d_n^{NS} a_s^n \right] \quad (1)$$

appeared in the literature [1]. Here  $R(s)$  is the well-known  $e^+e^-$  ratio,  $Q_F$  are the quarks charges,  $a_s = \alpha_s(Q^2)/\pi$  and  $\alpha_s(Q^2)$  is the  $\overline{\text{MS}}$ -scheme QCD coupling constant, which obeys the property of asymptotic freedom at large  $Q^2$ . The evaluation of  $d_4^{NS}$  [1] is the third step after analytical calculations of the  $\alpha_s^2$  [2] and  $\alpha_s^3$  corrections [3] [4] to the Adler function of vector currents. The expression for the  $\alpha_s^3$ -term was confirmed later on by really independent calculation of Ref. [5]. However, the first theoretical argument in favour of the validity of the result of Ref. [3] came from the foundation of Ref.[6], where it was shown that the product of the order  $\alpha_s^3$ -expression for the  $D^{NS}$ -function and of the similar approximation for the Bjorken polarized sum rule [7] is leading to the one-scale generalization of the quark-parton Crewther relation [8]. This generalized expression receives extra term, proportional to the two-loop QCD  $\beta$ -function [6]. The guess that this foundation will be correct in all orders of perturbation theory was made with caution in Ref. [6] and at more confidence level in Ref. [9]. Moreover, extra arguments in favour of relating this property to the effect of violation of the conformal symmetry of massless theory of strong interactions by the terms, proportional to the factor  $\beta(a_s)/a_s$ , were given in Ref.[9] in momentum space. Later on this property got more solid support after its detailed proof, performed in coordinate space [10]. In this letter I will show, how the application of the analog of the original Crewther relation [8] may help to get deeper understanding of the status of the 5-loop QCD result of Ref. [1] and of the part of its QED limit [11]. Note, that both these analytical expressions are giving rise to definite personal worries, which will be specified below. In view of this it seems urgent to propose concrete ways of their independent cross-check.

## 2 Formulation of the problems

The result of Ref. [1], namely Eq. (1), was presented in the case of  $SU(3)$  group only, without singling out the corresponding Casimir operators  $C_F$  and  $C_A$ . This does not allow one to study special theoretical features of  $\alpha_s^4$ -coefficients to both  $D^{NS}(Q^2)$  and to the photon vacuum polarization constant  $Z_{ph}$  in particular, which are manifesting themselves at the  $\alpha_s^3$ -level in the case of  $SU(N)$  group. Indeed, in Ref. [3] it was observed, that at the  $\alpha_s^3$ -level  $\zeta_3$ -term, which appears in  $Z_{ph}$  in QCD, is cancelling out in the case of  $SU(N)$  gauge group with  $C_A = C_F = T f/2 = N$ , i.e. in the case of the concrete renormalization group constant of  $SU(4)$  supersymmetric Yang-Mills theory, studied in detail at the three-loop level in Ref. [12]. This observation gave the authors of Ref. [16] some additional theoretical arguments in favour of the validity of the part of the obtained in this work 4-loop results. It will be highly desirable to get similar gentle support of the validity of 5-loop QCD expression of Ref. [1].

However, at present at this level there are extra unexplained theoretical questions. Indeed, let us have a look to the structure of interesting part of analytical result of Ref.[1], namely to the perturbative expression for the single-fermion contribution to the QED  $\beta$ -function (which is proportional to the single-fermion QCD contribution to Eq.(1). Its 5-loop expression was presented in Ref. [11] and has the following form:

$$\begin{aligned}\beta_{QED}^{[1]} &= \frac{4}{3}A + 4A^2 - 2A^3 - 46A^4 + \left(\frac{4157}{6} + 128\zeta_3\right)A^5 \\ &= \frac{4}{3}A \times C_D^{NS}(A)\end{aligned}\tag{2}$$

where  $A = \alpha/(4\pi)$  and  $\alpha$  is the QED coupling constant.

It can be shown that the coefficients of Eq.(2) are scheme-independent (see e.g. [13]), at least in the schemes, not related to the lattice regularization. This property is related to the conformal symmetry of the subsets of graphs, contributing to Eq.(2). In this limit the expansion parameter  $A$  is not running and is simply the constant (it does not depend from any scale). The analytical structure of the 5-loop result of Ref. [11] differs from the previously known terms: it contains  $\zeta_3$ -term in the 5-loop coefficient.

Note, that at the intermediate stages of calculations of the 3-loop correction to Eq.(2) [14], [15]  $\zeta_3$ -terms were appearing, but they cancelled out in the ultimate result. Moreover, in Ref. [15] this feature was related to the property of the conformal invariance of this part of QED  $\beta$ -function, though no proofs or references were given.

Next, in the process of evaluation of the 4-loop term in Eq.(2) [16] the contributions with two transcendentality  $\zeta_3$  and  $\zeta_5$  appeared at the intermediate stages of calculations, but these contributions cancelled in the final result.

At the five-loop level one may expect, that  $\zeta_3$ ,  $\zeta_5$  and  $\zeta_7$  should appear, but cancel down in the final result. However, Eq.(2) demonstrate that for  $\zeta_5$  and  $\zeta_7$  this property is valid, while for  $\zeta_3$  this is not the case!

Personally, I do not know any examples where the similar features, namely the cancellations of higher transcendentality, but appearance of lower ones in higher orders, despite their cancellation at lower orders, are manifesting themselves. I do not know whether this observation may be related to the un-proved property of “maximal transcendentality”, which at present is widely discussed while considering perturbative series for different quantities in the conformal invariant  $N = 4$  SYM theory (see e.g. [17], [18]). Thus we do not know whether the appearance of the transcendental term may be considered pro or contra the validity of the results of Refs. [11], [1].

In any case, to clarify the status of this new feature of perturbative series in QED it is highly desirable to get *independent calculational verification* of the results of Ref. [11], [1].

### 3 Proposed procedures of cross-checks

The study of the prediction of the coefficient before  $C_F^4 \alpha_s^4$  contribution to the perturbative QCD term in the Bjorken sum rule of the polarized charged lepton- polarized nucleon deep-inelastic scattering is one of the ways, which may allow to understand better the status of the results of Eq.(2). This sum rule can be defined as

$$\text{Bjp}(Q^2) = \int_0^1 \left[ g_1^{lp}(x, Q^2) - g_1^{ln}(x, Q^2) \right] dx = \frac{1}{6} g_A C_{Bjp}(a_s) = \frac{1}{6} g_A \left[ 1 + \sum_{n=1}^{n=4} c_n a_s^n \right] \quad (3)$$

Using the conformal-invariant (c-i) limit of the generalized Crewther relation, discover in Ref. [6], it is possible to write-down the following relation

$$C_{Bjp}(a_s(Q^2)) C_D^{NS}(a_s(Q^2))|_{c-i} = 1 \quad , \quad (4)$$

It follows from application of operator product expansion method for the three-point function of axial-vector-vector non-singlet quark currents in the momentum space [9] (for more details see [19]) and is reproducing original Crewther relation, obtained from the coordinate space considerations of Ref. [8] and Ref. [20] as well. Note also that Eq.(4) differs from the one, derived in Ref. [21] (for the related analysis see Ref.[22]). Indeed, in Eq.(4) the coupling constant  $a_s$  is **scale independent** and is defined in the Euclidean region.

Taking into account the results of previous QCD calculations and generalizing 5-loop result of Ref.[11] to the case of QCD in the conformal invariant limit, one has

$$C_D^{NS}(a_s) = \left[ 1 + \frac{3}{4} C_F a_s - \frac{3}{32} C_F^2 a_s^2 - \frac{69}{128} C_F^3 a_s^3 + \left( \frac{4157}{2048} + \frac{3}{8} \zeta_3 \right) C_F^4 a_s^4 \right] \quad . \quad (5)$$

where  $C_F = (N^2 - 1)/(2N)$  in the case of  $SU(N)$  gauge group. Using now Eq.(4) we get scheme-independent contributions to the Bjorken polarized sum rule, which include two new order  $\alpha_s^4$  terms <sup>1</sup>:

$$C_{Bjp}(a_s) = 1 - \frac{3}{4} C_F a_s + \frac{21}{32} C_F^2 a_s^2 - \frac{3}{128} C_F^3 a_s^3 - \left( \frac{4823}{2048} + \frac{3}{8} \zeta_3 \right) C_F^4 a_s^4 \quad (6)$$

The coefficients of order  $a_s$ ,  $a_s^2$  and  $a_s^3$ -terms are in agreement with the result of explicit calculations, performed in Refs.[24], [23] and [7] respectively. It should be also mentioned that the similar consideration was performed previously in Ref. [20] at the level of  $a_s$  corrections, but the  $a_s^2$ -term was not predicted there.

The direct evaluation of the **predicted**  $a_s^4$  coefficient may be rather useful for the independent cross-check of the QED results of Ref.[11] and thus of the related part of the QCD expression from Ref. [1]. This evaluation should clarify whether  $\zeta_3$  term is appearing in the  $a_s^4$  correction to  $C_{Bjp}(a_s)$  or not. This will give the most decisive argument pro or

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<sup>1</sup>It is possible to show that in the conformal invariant limit logarithmic QCD contributions to the Gross-Llewellyn Smith sum rule coincide with the ones for the Bjorken polarized sum rule in all orders of perturbation theory, see e.g. [19].

contra the validity of the  $\alpha_s^4$  results of Eq. (5), which are following from the ones of Eq.(2), presented in [11].

Note, that there are also at least two other possibilities for the cross-check of the result of Eq.(2). The first one is related to the extension to 5-loops of Dyson-Shwinger-Johnson motivated analysis, performed by Broadhurst [25] at the 4-loop level. The 5-loop extension of the work of Ref.[25], based on the calculations of definite 5-loop anomalous dimensions in QED from the 4-loop finite scheme-independent integrals, should demonstrate the cancellation of  $\zeta_5$  and  $\zeta_7$  terms and clarify whether  $\zeta_3$ - contribution is appearing or not.

Another way for checking the result of Eq.(2) may be based on the generalization of the Background Field Method to the case of 5-loop QED calculations. Note, however, that up to now this method was directly used at the 3-loop level only [26].

## 4 Conclusion

In this letter we address the question on the available at present possibilities of independent cross-checks of the part of the result of Ref. [1]. To our point of view the most decisive and urgent test may come from evaluation of the coefficient of  $C_F^4 \alpha_s^4$  contribution to the Bjorken sum rule, which may present additional arguments pro or contra the appearance of  $\zeta_3$ -term in the 5-loop perturbative correction of one-fermion loop contribution into the QED  $\beta$ -function.

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